TEMPERATURE FIELD IN A HOLLOW FINITE-LENGTH CYLINDER WITH AN ARBITRARILY MOVING HEAT SOURCE

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An analytical solution is obtained to the problem of heating a hollow cylinder from an arbitrary heat source. The asymptotic of the characteristic equation is derived and the eigenvalues of the corresponding Sturm–Liouville equation are tabulated.

We consider the problem of a temperature field $\theta(\rho, \varphi, \eta, \mathrm{Fo})$ in a hollow finite-length subjected to a heat source which moves in an arbitrary mode. The heat transfer at both the inside and the outside surface of this cylinder proceeds according to Newton's law, while the end surfaces are thermally insulated.

The problem reduces to integrating the heat transfer equation

$$\frac{\partial\theta}{\partial \mathrm{Fo}} = \frac{\partial^2\theta}{\partial\rho^2} + \frac{1}{\rho}\cdot\frac{\partial\theta}{\partial\rho} + \frac{1}{\rho^2}\cdot\frac{\partial^2\theta}{\partial\varphi^2} + (k\pi)^2\frac{\partial^2\theta}{\partial\eta^2} + \mathrm{Po}\, f_{\mathrm{v}}(\rho, \varphi, \eta, \mathrm{Fo}) \tag{1}$$

with initial and boundary conditions

$$\theta(\rho, \varphi, \eta, 0) = 1, \tag{2}$$

$$\left(\frac{\partial\theta}{\partial\rho} - \mathrm{Bi}_1\theta\right)_{\rho=1} = \mathrm{Ki}\, f_{\mathrm{p}}(\varphi, \eta, \mathrm{Fo}), \tag{3}$$

$$\left(\frac{\partial\theta}{\partial\rho} + \frac{1}{\rho_0}\mathrm{Bi}_2\theta\right)_{\rho=\rho_0} = \frac{1}{\rho_0}\mathrm{Bi}_2, \tag{4}$$

$$\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} = \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=\pi} = 0, \tag{5}$$

$$\theta(\rho, \varphi, \eta, \mathrm{Fo})_{\varphi=0} = \theta(\rho, \varphi, \eta, \mathrm{Fo})_{\varphi=2\pi}. \tag{6}$$

System (1)-(6) appears in dimensionless form. The introduction of an internal heat source $(\mathrm{Po})f_{\mathrm{v}}(\rho, \varphi, \eta, \mathrm{Fo})$ generalizes the problem.

The choice of functions $f_{\mathrm{p}}(\varphi, \eta, \mathrm{Fo})$ and $f_{\mathrm{v}}(\rho, \varphi, \eta, \mathrm{Fo})$ is dictated by the trajectory, the velocity, and by the space-time distribution of the source of thermal flux density.

Using the method of finite integral transformations [1, 2] and the Duhamel theorem [3], we obtain the solution:

$$\theta(\rho, \varphi, \eta, \mathrm{Fo}) = \theta_1(\rho, \mathrm{Fo}) + \mathrm{Ki}\frac{\partial}{\partial \mathrm{Fo}}\int_0^{\mathrm{Fo}}\int_0^{2\pi}\int_0^{\pi} f_{\mathrm{p}}(\xi, \zeta, \lambda)$$
$$\times\theta_{2G}(\rho, \varphi, \xi, \eta, \zeta, \mathrm{Fo}-\lambda)\, d\xi d\zeta d\lambda + \mathrm{Po}\frac{\partial}{\partial \mathrm{Fo}}\int_0^{\mathrm{Fo}}\int_0^{2\pi}\int_0^{\pi}\int_1^{\rho_0} f_{\mathrm{v}}(\gamma, \xi, \tag{7}$$
$$\zeta, \lambda)\,\theta_{1G}(\rho, \gamma, \varphi, \xi, \eta, \zeta, \mathrm{Fo}-\lambda)\, d\gamma d\xi d\zeta d\lambda,$$

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where

$$\theta_1(\rho,\ \mathrm{Fo}) = \frac{\frac{1}{\mathrm{Bi}_1} + \ln\rho}{\frac{1}{\mathrm{Bi}_1} + \frac{1}{\mathrm{Bi}_2} + \ln\rho_0} + \pi\,\mathrm{Bi}_1 \sum_{\mu} A_\mu V_0(\mu\rho)\exp(-\mu^2\mathrm{Fo}). \tag{8}$$

It can be shown that $\theta_{2G}(\rho, \varphi, \xi, \eta, \zeta, \mathrm{Fo})$ is a Green function of the second kind [4] for problem (1)-(6) with Po = 0, the explicit expression for which is

$$\theta_{2G}(\rho,\ \varphi,\ \xi,\ \eta,\ \zeta,\ \mathrm{Fo}) = -\frac{1}{4\pi}\sum_{m,\mu,} \mu^2 A_{\mu m} V_m(\mu\rho)\exp\left[-im(\varphi-\xi)\right]$$

$$\times \int_0^{\mathrm{Fo}} \exp(-\mu^2\tau)\left\{\vartheta\left(\frac{\eta+\zeta}{2\pi},\ k^2\pi\tau\right) + \vartheta\left(\frac{\eta-\zeta}{2\pi},\ k^2\pi\tau\right)\right\} d\tau, \tag{9}$$

with $\vartheta(x, t)$ denoting the Jacobi theta function [3] and

$$A_{\mu m} = \left\{\frac{\pi^2}{4}\left(\mu^2\rho_0^2 + \mathrm{Bi}_2^2 - m^2\right)V_m^2(\mu\rho_0) - \left(\mu^2 + \mathrm{Bi}_1^2 - m^2\right)\right\}^{-1}. \tag{10}$$

An analysis of (7) will readily show that, if

$$f_V(\rho,\ \varphi,\ \eta,\ \mathrm{Fo}) = \delta\left[\rho - (1+\Delta\rho)\right] f_p(\varphi,\ \eta,\ \mathrm{Fo}),$$

i.e., if the internal heat source is located directly at the inside surface of the cylinder (which corresponds to a delta-type distribution of thermal flux density along coordinate ρ at point $1 + \Delta\rho$), then the effect of each element acting independently is almost concurrent when Po = Ki.

Summation over μ in (9) is carried out over all positive roots of the equation

$$\left\{\frac{d\ln V_m(\mu\rho)}{d\rho}\right\}^{-1}_{\rho=\rho_0} + \frac{\rho_0}{\mathrm{Bi}_2} = 0, \tag{11}$$

$$V_m(\mu\rho) = \left\{(m-\mathrm{Bi}_1)Y_m(\mu) - \mu Y_{m+1}(\mu)\right\} J_m(\mu\rho) + \left\{\mu J_{m+1}(\mu) - (m-\mathrm{Bi}_1)J_m(\mu)\right\} Y_m(\mu\rho) \tag{12}$$

where (12) is the kernel of a Hankel transform.

The roots of Eq. (11) are listed in Table 1 for various values of ρ_0, m, and $\mathrm{Bi}_1 = (1/\rho_0)\mathrm{Bi}_2$.

With the aid of respective asymptotic formulas [5]

$$J_m(\mu) = \sqrt{\frac{2}{\pi\mu}}\cos\left[\mu - \frac{\pi}{2}\left(m+\frac{1}{2}\right)\right],\quad J_{m+1}(\mu) = \sqrt{\frac{2}{\pi\mu}}\cos\left[\mu - \frac{\pi}{2}\left(m+\frac{3}{2}\right)\right], \tag{13}$$

$$Y_m(\mu) = \sqrt{\frac{2}{\pi\mu}}\sin\left[\mu - \frac{\pi}{2}\left(m+\frac{1}{2}\right)\right],\quad Y_{m+1}(\mu) = \sqrt{\frac{2}{\pi\mu}}\sin\left[\mu - \frac{\pi}{2}\left(m+\frac{3}{2}\right)\right], \tag{14}$$

which are valid for $|\mu| \gg 1$ and $|\mu| \gg m$, we obtain the following asymptotic representation:

$$V_m(\mu\rho) \approx \frac{2}{\pi\mu\sqrt{\rho}}\left\{(\mathrm{Bi}_1 - m)\sin\mu(\rho-1) + \mu\cos\mu(\rho-1)\right\}. \tag{15}$$

Considering that the characteristic equation (11) can be made to appear as

$$\frac{V_m(\mu\rho_0)}{V_{m1}(\mu\rho_0)} = \frac{\mu\rho_0}{\mathrm{Bi}_2}, \tag{16}$$

where

$$V_{m1}(\mu\rho) = -\frac{dV_m(\mu\rho)}{\mu d\rho}, \tag{17}$$

TABLE 1. Values of the Roots of Eq. (11)

μ	Bi_1 = 0,5	1	2	3	5	7,5	10
				$\rho_0 = 1{,}1;\ m = 0$			
μ_1	3,149	4,435	6,221	7,557	9,602	11,531	13,066
μ_2	31,741	32,050	32,649	33,227	34,318	35,579	36,738
				$\rho_0 = 1{,}1;\ m = 1$			
μ_1	3,342	4,536	6,294	7,618	9,648	11,569	13,090
μ_2	31,755	32,064	32,663	33,240	34,331	35,592	36,751
				$\rho_0 = 1{,}04;\ m = 0$			
μ_1	4,992	7,047	9,934	12,126	15,552	18,893	21,640
μ_2	78,750	79,250	80,000	80,054	81,507	84,006	84,501
				$\rho_0 = 1{,}04;\ m = 1$			
μ_1	5,086	7,115	9,982	12,166	15,582	18,920	21,662
				$\rho_0 = 1{,}02;\ m = 0$			
μ_1	7,064	9,983	14,095	17,235	22,175	27,050	31,107
μ_2	157,5	157,7	158,5	159,0	160,0	161,5	163,0
				$\rho_0 = 1{,}02;\ m = 1$			
μ_1	7,133	10,030	14,130	17,263	22,197	27,065	31,120

we have the following asymptotic approximation to Eq. (16):

$$\operatorname{tg}\mu(\rho_0-1)=\frac{(\mathrm{Bi}_1-m)+\frac{1}{\rho_0}(\mathrm{Bi}_2+m)}{\mu^2-\frac{1}{\rho_0}(\mathrm{Bi}_2+m)(\mathrm{Bi}_1-m)}\,\mu, \tag{18}$$

which together with (15) and (18) yields

$$V_m^2(\mu\rho_0)=\frac{4\rho_0}{\pi^2}\cdot\frac{(\mathrm{Bi}_1-m)^2+\mu^2}{(\mathrm{Bi}_2+m)^2+\mu^2\rho_0^2}, \tag{19}$$

and, consequently,

$$A_{\mu m}V_m(\mu\rho)\sim\frac{1}{\mu^2}. \tag{20}$$

Solution (7) is quite general and provides a tool for analyzing a large class of various heat sources. In order to arrive at solutions which will be of practical value, however, it becomes necessary to perform a triple integration. This difficulty can be overcome in some cases when the thermal flux density is distributed according to certain laws.

In [6, 7] the thermal flux density was defined by a Gaussian distribution. Accordingly, letting

$$f_{\Pi}(\varphi,\ \eta,\ \mathrm{Fo})=\frac{1}{2\pi\sigma^2}\exp\left[-\frac{(\varphi-\mathrm{Pe}_\omega\mathrm{Fo})^2+(\eta-\mathrm{Pe}_\upsilon\mathrm{Fo})^2}{2\sigma^2}\right], \tag{21}$$

with Po = 0 and replacing (7) by its equivalent

$$\theta(\rho,\ \varphi,\ \eta,\ \mathrm{Fo})=\theta_1(\rho,\ \mathrm{Fo})-\frac{\mathrm{Ki}}{2\pi}\sum_{m,\mu,q}\mu^2A_{\mu m}V_m(\mu\rho)$$
$$\times\exp(-im\varphi)\cos q\eta\exp\{-[\mu^2+(k\pi)^2q^2]\,\mathrm{Fo}\}$$
$$\times\int_0^{\mathrm{Fo}}\int_0^{2\pi}\int_0^{\pi}f_{\mathrm{P}}(\xi,\ \zeta,\ \tau)\exp\{[\mu^2+(k\pi)^2q^2]\,\tau+im\xi\}\cos q\zeta\,d\xi\,d\zeta\,d\tau, \tag{22}$$

we can evaluate the triple integral in solution (22) as follows:

$$\int_0^{\mathrm{Fo}}\int_0^{2\pi}\int_0^{\pi}f_{\mathrm{P}}(\xi,\ \zeta,\ \tau)\exp\{[\mu^2+(k\pi)^2q^2]\,\tau+im\xi\}\cos q\zeta\,d\xi\,d\zeta\,d\tau$$
$$=\int_0^{\mathrm{Fo}}\exp\{[\mu^2+(k\pi)^2q^2]\,\tau\}\left\{\frac{1}{\sigma\sqrt{2\pi}}\int_0^{2\pi}\exp\left[-\frac{(\xi-\mathrm{Pe}_\omega\tau)^2}{2\sigma^2}\right]\exp(im\xi)\,d\xi\right\}$$

$$\times\left\{\frac{1}{\sigma\sqrt{2\pi}}\int_0^{\pi}\exp\left[-\frac{(\zeta-\mathrm{Pe}_v\tau)^2}{2\sigma^2}\right]\cos q\zeta d\zeta\right\}d\tau$$

$$=\int_0^{\mathrm{Fo}}\exp\{[\mu^2+(k\pi)^2q^2]\tau\}\exp(im\,\mathrm{Pe}_\omega\tau)\cos q(\mathrm{Pe}_v\tau)$$

$$\times\left\{\exp\left[-\frac{1}{2}\sigma^2(m^2+q^2)\right]+o(\sigma)\right\}. \tag{23}$$

The following formulas have been used here:

$$\int_0^{\infty}\exp(-\beta x^2-\gamma x)\cos bxdx=\frac{1}{4}\sqrt{\frac{\pi}{\beta}}\left\{\exp\frac{(\gamma-ib)^2}{4\beta}\right.$$

$$\left.\times\left[1-\Phi\left(\frac{\gamma-ib}{2\sqrt{\beta}}\right)\right]+\exp\frac{(\gamma+ib)^2}{4\beta}\left[1-\Phi\left(\frac{(\gamma+ib)}{2\sqrt{\beta}}\right)\right]\right\}, \tag{24}$$

$$\int_0^{\infty}\exp(-\beta x^2-\gamma x+ibx)\,dx=\frac{1}{2}\sqrt{\frac{\pi}{\beta}}\exp\frac{(\gamma-ib)^2}{4\beta}\left[1-\Phi\left(\frac{\gamma-ib}{2\sqrt{\beta}}\right)\right] \tag{25}$$

and the asymptotic expansion for $|z|\gg 1$, $-(3/4)\pi+\varepsilon<\arg z<(3/4)\pi-\varepsilon\ (\varepsilon>0)$:

$$\frac{\sqrt{\pi}}{2}\left[1-\Phi(z)\right]\approx\frac{\exp(-z^2)}{2z}\left[1-\frac{1}{2z^2}+\frac{3}{4z^4}-\dots\right], \tag{26}$$

where $\Phi(z)=(2/\pi)\int_0^z\exp(-x^2)dx$ is the probability integral.

In this way, the temperature field due to a heat source of the form (21) is described, within an accuracy down to $o(\sigma)$, by the expression

$$\theta(\rho,\ \varphi,\ \eta,\ \mathrm{Fo})=\theta_1(\rho,\ \mathrm{Fo})-\frac{\mathrm{Ki}}{2\pi}\sum_{m,\mu,q}\mu^2A_{\mu m}V_m(\mu\rho)\exp(-im\varphi)$$

$$\times\cos(q\eta)\exp\{-[\mu^2+(k\pi)^2q^2]\,\mathrm{Fo}\}\int_0^{\mathrm{Fo}}\{\exp(im\,\mathrm{Pe}_\omega\tau)$$

$$\times\cos q(\mathrm{Pe}_v\tau)\exp\{[\mu^2+(k\pi)^2q^2]\tau\}\left\{\exp\left[-\frac{1}{2}\sigma^2(m^2+q^2)\right]\right\}d\tau. \tag{27}$$

Considering that the temperature distribution produced by a heat source of the form

$$f_p(\varphi,\ \eta,\ \mathrm{Fo})=\delta(\varphi-\mathrm{Pe}_\omega\mathrm{Fo})\,\delta(\eta-\mathrm{Pe}_v\mathrm{Fo}), \tag{28}$$

can be expressed as

$$\theta(\rho,\ \varphi,\ \eta,\ \mathrm{Fo})=\theta_1(\rho,\ \mathrm{Fo})-\frac{\mathrm{Ki}}{2\pi}\sum_{m,\mu,q}\mu^2A_{\mu m}V_m(\mu\rho)$$

$$\times\exp(-im\varphi)\cos(q\eta)\exp\{-[\mu^2+(k\pi)^2q^2]\,\mathrm{Fo}\}$$

$$\times\int_0^{\mathrm{Fo}}\exp\{[\mu^2+(k\pi)^2q^2]\tau\}\exp(im\,\mathrm{Pe}_\omega\tau)\cos q(\mathrm{Pe}_v\tau)\,d\tau, \tag{29}$$

we now examine the difference between solutions (27) and (29):

$$\Delta\theta(\rho,\ \varphi,\ \eta,\ \mathrm{Fo},\ \sigma^2)=\frac{\mathrm{Ki}}{2\pi}\sum_{m,\mu,q}\mu^2A_{\mu m}V_m(\mu\rho)\exp(-im\varphi)$$

$$\times\cos(q\eta)\exp\{-[\mu^2+(k\pi)^2q^2]\,\mathrm{Fo}\}\int_0^{\mathrm{Fo}}\exp(im\,\mathrm{Pe}_\omega\tau)$$

$$\times \cos q\,(\mathrm{Pe}_v\,\tau)\left\{1-\exp\left[-\frac{1}{2}(m^2+q^2)\,\sigma^2\right]\right\}$$
$$\times \exp\left\{[\mu^2+(k\pi)^2 q^2]\,\tau\right\} d\tau. \quad (30)$$

Since the calculations will be sufficiently accurate at some $m = m_0$ and $q = q_0$, we have

$$|\Delta\theta(\rho,\ \varphi,\ \eta,\ \mathrm{Fo},\ \sigma^2)| \leqslant \mathrm{Ki}\left\{1-\exp\left[-\frac{1}{2}(m_0^2+q_0^2)\,\sigma^2\right]\right\}$$
$$\times M_0(\rho,\ \varphi,\ \eta,\ \mathrm{Fo}) = M(\rho,\ \varphi,\ \eta,\ \mathrm{Fo}), \quad (31)$$

where the expression for M_0 can be easily derived from (30).

Thus, the difference will not exceed a prescribed small magnitude $M(\rho, \varphi, \eta, \mathrm{Fo})$, if the inequality

$$\sigma^2 \leqslant \frac{2}{m_0^2+q_0^2}\ln\frac{1}{1-\dfrac{M}{\mathrm{Ki}\,M_0}} \quad (32)$$

holds true.

Condition (32) defines the dispersion range within which the temperature field to a Gaussian heat source will be approximated with a given error corresponding to the respective Green function.

The results given in [9-12] may be viewed as special cases of solution (7).

NOTATION

θ, ρ, η	is the dimensionless temperature and space coordinates;
Bi_1, Bi_2	is the Biot number;
Fo	is the Fourier number;
Ki	is the Kirpichev number;
Po	is the Pomerantsev number;
α_1, α_2	is the coefficients of internal and external heat transfer respectively;
R_1, R_2	is the inside and outside radii of a hollow cylinder respectively;
λ	is the thermal conductivity;
a	is the thermal diffusivity;
T_c, T_0	is the ambient temperatures;
k	is the cylinder radius-to-length ratio;
Pe_ω	is the dimensionless angular velocity;
Pe_v	is the dimensionless velocity along the η-axis;
$J_m(\mu\rho)$, $Y_m(\mu\rho)$	is the Bessel functions;
$\delta(\varphi-\varphi_0)$, $\delta(\eta-\eta_0)$	is the delta function.

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